IC/87/205

INTERNATIONAL ATOMIC ENERGY AGENCY

UNITED NATIONS EDUCATIONAL, SCIENTIFIC AND CULTURAL ORGANIZATION

INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

HYPER-KAHLER METRICS AND HARMONIC SUPERSPACE

A.E. Hassouni

T. Lhallabi

E.G. Oudrhiri-Safiani

and

E.H. Saidi

1987 MIRAMARE-TRIESTE

IC/87/205

International Atomic Energy Agency
and
United Nations Educational Scientific and Cultural Organization

INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

HYPER-KAHLER METRICS AND HARMONIC SUPERSPACE *

A.E. Hassouni ** and T. Lhallabi **
International Centre for Theoretical Physics, Trieste, Italy

E.G. Oudrhiri-Safiani
Laboratoire de Physique Theorique, Faculte des Sciences, Av. Ibn Battouta, P.O. Box 1014, Rabat, Morocco

and

E.H. Saidi **
International Centre for Theoretical Physics, Trieste, Italy.

MIRAMARE - TRIESTE
August 1987

* To be submitted for publication.

** Permanent address: Laboratoire de Physique Théorique, Faculté des Sciences, Av. Ibn Battouta, P.O. Box 1014, Rabat, Morocco.

ABSTRACT

Using the harmonic superspace techniques with central charges, we study the U(1) hypermultiplet self-interaction by working out a general class of solutions of the auxiliary field equations of motion in presence of spinor fields. These solutions depend on an extra unfixed degree of freedom ω which can be gauged away through an additional condition. The resulting metric is ω-dependent and is shown to be hyper-Kähler. The induced scalar potential is discussed. The fermionic solutions are found to be rotated by an extra local SU(2) symmetry leading to the normalization given by Galperin et al. for the vielbein fields. The full spinorial contribution of the model is given in terms of the N=2 on shell multiplet $(f^i, \xi, \bar{\eta})$.

1. INTRODUCTION

Supersymmetric non-linear σ models provide an important setting for the study of the properties of supersymmetric theories. The examination of these theories led to the discovery of fascinating relations such as the connection between the Kähler geometry and $N=1$, $d=4$ matter superfield [1]. In four-dimensional space-time, one supersymmetry requires a Kähler manifold and two supersymmetries require a hyper-Kähler one [2,3]. Moreover, two-dimensional supersymemtric non-linear σ-models are of much interest today because of their connection with string theory [4]. The later is assumed to be the consistent framework for unification of all known forces of nature [4,5].

Recently, there has been some progress in the study of $N=2$, $d=4$ ($N=4$, $d=2$) supersymmetric σ-models [2,3,6,7] and above all the use of harmonic superspace techniques [8-12]. Using this formalism, Galperin et al. (referred hereafter to as GIOS) showed that the bosonic part of the U(1) non-renormalizable $N=2$ Fayet-Sohnius (F-S) matter multiplet self-coupling describes a $N=2$ hyper-Kahler non-linear σ-model with a TUB-NUT metric [9]. Their procedure consists of solving the non-linear auxiliary field equations of motion taking advantage of the global U(1) symmetry exhibited by the model.

In this paper, we study the two following points. First, we consider the bosonic U(1) self-interacting (F-S) supefield model with central charges and derive a class of solutions of the auxiliary field equations more general than those given in Ref.[9]. These solutions depend on an extra degree of freedom ω which can be gauged away throughout an additional condition. Then we derive the resulting generalized metric and check its hyper-Kähler. Second, we examine the fermionic contribution by solving the full set of auxiliary equations of motion in presence of spinor fields. Finally, we work out the complete $N=2$ supersymmetric Lagrangian in terms of the $N=2$ on shell multiplet degrees of freedom f^i, ξ, η and their conjugates $\bar{f}_i, \bar{\xi}$ and $\bar{\eta}$.

The presentation is as follows. In Sec.2, we consider the U(1) self-interacting hypermultiplet action with central charges (for simplicity, only one central charge is discussed) from which we derive the equations of motion of the auxiliary fields. Then, we give a general class of solutions of these non-linear differential equations for the pure bosonic case. These solutions are discussed and turn out to depend in general on the central charge. The corresponding generalized metric is worked out. In Sec.3, we use the superfield formalism to prove the hyper-Kähler structure of our metric by constructing a real SU(2) triplet $\Omega^{(ij)}$ of closed two forms. The vielbein normalization given by Galperin et al. is shown to be unchanged. In Sec.4, we turn to the fermionic

contribution. A class of solutions of the auxiliary field equations of motion in presence of spinor fields is derived. For the fermionic fields, we find that the solutions depend on the bosonic fields f^i and $\bar{f}^i$ as well as on two arbitrary different Weyl spinors ξ and η independent of the harmonic variables. These solutions are rotated by a local SU(2) transformation leading to:

1. the right Dirac Lagrangian coefficients without need to make any perturbative expansion in the coupling parameter λ;
2. a normalization of the spinor fields similar to the vielbein one of Sec.3.

For the bosonic auxiliary fields, there is also a spinorial contribution to their solutions. Finally, we give the full Lagrangian in terms of the $N=2$ on shell multiplet degrees of freedom (f^i, ξ, η). The last section is devoted to conclusion.

2. THE GEOMETRICAL STRUCTURE OF THE SELF-INTERACTING HYPERMULTIPLET ACTION

In this section, we give a generalization of the hyper-Kähler GIOS metric [9], by working out a more general solutions of the auxiliary field equations of motion of the (F-S) superfield ϕ^+. But first of all, let us recall the action describing the hypermultiplet self-interation with central charges:

$$
\begin{aligned}
S &= \int d^4x_A\, du\, L \\
L &= \frac{1}{2}\int_{\mathcal{J}} d^4\theta^+ \frac{dz}{R}\, \mathcal{L} \qquad\qquad (2.1)\\
\mathcal{L} &= \bar{\tilde{\Phi}}^+ D^{++} \tilde{\Phi}^+ - \tilde{\Phi}^+ \tilde{D}^{++} \bar{\tilde{\Phi}}^+ + \lambda\left(\bar{\tilde{\Phi}}^+ \Phi^+\right)^2 .
\end{aligned}
$$

The extra dimension z realizes the unique central charge considered here. $\mathcal{J}$ is a linear symmetric domain of length R which has to be viewed as a regulator parameter [13]. The generalized (F-S) analytic superfield $\tilde{\phi}^+$ is chosen as

$$\tilde{\phi}^+ = \phi^+ \exp(-imz) \quad , \qquad (2.2)$$

where m is the central charge eigenvalue and the analytic superfield ϕ^+ read as:

$$\Phi^{+} = \varphi^{+} + \sqrt{2}\,\theta^{+}\psi + \sqrt{2}\,\bar{\theta}^{+}\bar{\chi} + \theta^{+2}F^{-} + \bar{\theta}^{+2}G^{-} + i\,\theta^{+}\sigma^{\mu}\bar{\theta}^{+}A^{-}_{\mu}$$
$$+ \sqrt{2}\,\theta^{+2}\bar{\theta}^{+}\bar{\gamma}^{(--)} + \sqrt{2}\,\bar{\theta}^{+2}\theta^{+}\delta^{(--)} + \theta^{+2}\bar{\theta}^{+2}\Delta^{(--)} . \qquad (2.3)$$

The introduction of this central charge is dictated by the two following reasons. 1. To have an insight on the scalar potential in terms of the component fields of the Lagrangian (2.1) i.e. its form and whether it has non-trivial extrema. Note that in absence of central charge the scalar potential is identically zero. 2. Since the N=2 global multiplet $(0^4, 1/2^2)$ is central charge dependent, the later should play a certain role in deriving the general solutions of the auxiliary field equations.

The generalized harmonic derivative is given by:

$$\tilde{D}^{++} = D^{++} + i\,(\theta^{+2} - \bar{\theta}^{+2})\,\frac{\partial}{\partial z} \qquad (2.4)$$

$$D^{++} = \partial^{++} - 2i\,\theta^{+}\sigma^{\mu}\bar{\theta}^{+}\partial_{\mu} . \qquad (2.5)$$

The equation of motion in terms of the superfield $\tilde{\phi}^{+}$ is obtained from Eq.(2.1) and reads as

$$\tilde{D}^{++}\tilde{\phi}^{+} + \lambda(\tilde{\bar{\phi}}^{+}\tilde{\phi}^{+})\tilde{\phi}^{+} = 0 . \qquad (2.6)$$

Or equivalently in terms of the component fields of (2.3) by giving only the relevant equations for our study

$$(\partial^{++} + \lambda\bar{\varphi}^{+}\varphi^{+})\,\varphi^{+} = 0 \qquad (1)$$

$$(\partial^{++} + \lambda\bar{\varphi}^{+}\varphi^{+})\,\psi + \lambda\varphi^{+}(\bar{\varphi}^{+}\psi - \varphi^{+}\chi) = 0 \qquad (2)$$

$$(\partial^{++} + \lambda\bar{\varphi}^{+}\varphi^{+})\,\bar{\chi} + \lambda\varphi^{+}(\bar{\varphi}^{+}\chi + \varphi^{+}\bar{\psi}) = 0 \qquad (3)$$

$$(\partial^{++} + \lambda\bar{\varphi}^{+}\varphi^{+})\,F^{-} + \lambda\varphi^{+}(\bar{\varphi}^{+}F^{-} + \varphi^{+}\bar{G}^{-} + 2\psi\chi) - \lambda\bar{\varphi}^{+}\psi\psi + m\varphi^{+} = 0 \qquad (4)$$

$$(\partial^{++} + \lambda\bar{\varphi}^{+}\varphi^{+})\,G^{-} + \lambda\varphi^{+}(\bar{\varphi}^{+}G^{-} + \varphi^{+}\bar{F}^{-} - \bar{\psi}\bar{\chi}) - \lambda\bar{\varphi}^{+}\bar{\chi}\bar{\chi} - m\varphi^{+} = 0 \qquad (5)$$

$$(\partial^{++} + \lambda\bar{\varphi}^{+}\varphi^{+})\,A^{-}_{\mu} + \lambda\varphi^{+}(\bar{\varphi}^{+}A^{-}_{\mu} + \varphi^{+}\bar{A}^{-}_{\mu} - 2iJ_{\mu}) - 2i\,\bar{\varphi}^{+}H_{\mu} - 2\partial_{\mu}\varphi^{+} = 0 \qquad (6)$$

$$(2.7)$$

where

$$J^{\mu} = \psi\sigma^{\mu}\bar{\psi} - \chi\sigma^{\mu}\bar{\chi}$$

$$H^{\mu} = \psi\sigma^{\mu}\bar{\chi} . \qquad (2.8)$$

Eqs.(2.7) should be viewed as contraints on the infinite harmonic series of the component fields $\mathcal{C}$ in Eq.(2.3) of mass dimension 1, 3/2 and 2:

$$\mathcal{C}^{q>0}_{(x_A,u)} = \sum_{n=0}^{\infty} \mathcal{C}^{(i_1 \ldots i_{n+q}\, j_1 \ldots j_n)}_{(x_A)}\, u^{+}_{i_1} \ldots u^{+}_{i_{n+q}}\, u^{-}_{j_1} \ldots u^{-}_{j_n} . \qquad (2.9)$$

Before examining the solutions of the constraints (2.7), let us give the Lagrangian (2.1) in terms of the component fields

$$L = \frac{1}{2}\Big\{ (\bar{A}^{-}_{\mu}\partial^{\mu}\varphi^{+} - A^{-}_{\mu}\partial^{\mu}\bar{\varphi}^{+}) - 2i(\partial_{\mu}\psi\sigma^{\mu}\bar{\psi} - \chi\sigma^{\mu}\partial_{\mu}\bar{\chi})$$
$$+ 2m\left[\varphi^{+}(\bar{F}^{-} - \bar{G}^{-}) - \bar{\varphi}^{+}(F^{-} - G^{-}) + (\psi\chi + \bar{\psi}\bar{\chi})\right]$$
$$- i\lambda J^{\mu}(\bar{\varphi}^{+}A^{-}_{\mu} + \varphi^{+}\bar{A}^{-}_{\mu} + \tfrac{1}{2}J_{\mu}) + i\lambda\left[\bar{\varphi}^{+}H^{\mu}\bar{A}^{-}_{\mu} - \varphi^{+}\bar{H}^{\mu}A^{-}_{\mu}\right]$$
$$- 2\lambda\bar{\varphi}^{+}(F^{-}\bar{\psi}\bar{\chi} - G^{-}\psi\chi) + 2\lambda\varphi^{+}(\bar{F}^{-}\psi\chi - \bar{G}^{-}\bar{\psi}\bar{\chi})$$
$$- \lambda\varphi^{+}(F^{-}\bar{\psi}\bar{\psi} + G^{-}\chi\chi) - \lambda\bar{\varphi}^{+}(\bar{F}^{-}\psi\psi + \bar{G}^{-}\bar{\chi}\bar{\chi}) \Big\} . \qquad (2.10)$$

Note that this Lagrangian and its superfield form (2.1) have a global U(1) symmetry

$$\Phi^{+\prime} = e^{i\alpha}\Phi^{+}, \quad \overset{*}{\bar{\Phi}}{}^{+\prime} = e^{-i\alpha}\overset{*}{\bar{\Phi}}{}^{+}, \quad D^{++}\alpha = 0 \tag{2.11}$$

leading to the conserved Nother current N^{++}:

$$N^{++} = \overset{*}{\bar{\Phi}}{}^{+}\Phi^{+}, \quad D^{++}N^{++} = 0 \tag{2.12}$$

This U(1) invariance will play an essential role in solving the <u>non-linear differential</u> equations (2.7)

In what follows, we fix our attention on the pure bosonic part of Eqs.(2.7) and (2.10), i.e. ψ and χ set equal to zero, and work out the more general solutions of Eqs.(2.7) with one central charge. The U(1) symmetry Eqs.(2.11)-(2.12) allows us to write the field φ^+ in terms of the isodoublet field f^i and $\bar{f}^i$ as

$$\varphi^{+}(x_A, u^{\pm}) = f^{+}(x_A)\exp\left[-\lambda\beta(f, u^{\pm})\right] \qquad (1) \qquad (2.13)$$

Similarly, the U(1) invariance and the dimensional arguments lead us to make the following change of variables:

$$F^{-}(x_A, u^{\pm}) = t^{-}(f^i, m, \lambda, u^{\pm})\cdot\exp[\lambda\beta] \qquad (2)$$

$$G^{-}(x_A, u^{\pm}) = s^{-}(f^i, m, \lambda, u^{\pm})\cdot\exp[-\lambda\beta] \qquad (3) \qquad (2.13)$$

$$A^{-}_{\mu}(x_A, u^{\pm}) = B^{-}_{\mu}(f^i, \partial_\mu f^i, m, \lambda, u^{\pm})\cdot\exp[-\lambda\beta] \qquad (4)$$

Putting Eq.(2.13.1) in Eq.(2.7.1) constraints β to satisfy the following harmonic differential equation:

$$\partial^{++}\beta - f^{+}\bar{f}^{+} = 0 \;;\; \beta = -\overset{*}{\bar{\beta}}$$
$$f^{\pm} = f^{i}u^{\pm}_{i} \;;\; \bar{f}^{\pm} = \bar{f}^{i}u^{\pm}_{i} \tag{2.14}$$

Eq.(2.14) may be inverted and a solution more general than that of Ref.[9] reads as

$$\beta = \frac{1}{2}\left[f^{+}\bar{f}^{-} + f^{-}\bar{f}^{+}\right] + i\, C(f,\bar{f})\left[f^{+}\bar{f}^{-} - f^{-}\bar{f}^{+}\right] \tag{2.15}$$

with $C(f,\bar{f})$ an arbitrary dimensionless real function independent of the harmonic variables $u^{\pm}{}_{i}$

$$\partial^{++}C(f,\bar{f}) = 0 \;;\; \overset{*}{\bar{C}}(f,\bar{f}) = C(f,\bar{f}) \tag{2.16}$$

Before going ahead, we would like to note that one could drop out the extra piece of Eq.(2.15) and work out the corresponding non-linear σ model. We shall refer to this choice here below as the "GIOS gauge". So far, we shall keep C as an extra degree of freedom which can be gauged away by imposing a "scalar gauge condition".

Eq.(2.16) restricts $C(f,\bar{f})$ to have the following dependence on the fields f^i and $\bar{f}_i$ if space-time derivatives are ignored:

$$C(f,\bar{f}) = C(\rho^2)$$
$$\rho^2 \equiv f^{+}\bar{f}^{-} - f^{-}\bar{f}^{+} = f^{[+}\bar{f}^{-]} = f^{i}\bar{f}_{i} \tag{2.17}$$

We shall make use of Eq.(2.15) in the following form

$$\beta = \frac{1}{2} f^{(+}\bar{f}^{-)} + i\,\omega(\rho^2)$$
$$\overset{*}{\bar{\omega}}(\rho^2) = \omega(\rho^2) \;;\; [\omega] = 2 ;\; \partial^{++}\omega = 0 . \tag{2.18}$$

where (,) and [,] are respectively symmetrization and antisymmetrization symbols.

Moreover, it is interesting to note that in absence of dimensional parameters other than the field f^i, $\omega(\rho^2)$ is restricted to be proportional to ρ^2. However, when dimensional parameters such as the coupling λ and the central charge eigenvalue m are taken into account, the function ω turns out to have more freedom namely

$$\lambda\omega = \mathrm{funct}\left(\lambda\rho^2, \left(\frac{m^2}{\rho^2}\right)\right) . \tag{2.19}$$

Eq.(2.19) tells us that the general solution of Eq.(2.7) are central charge dependent as mentioned earlier.

Putting now Eqs.(2.13) into (2.7) and making use of Eq.(2.12), one can work out the unknown quantities of Eqs.(2.13). For the t^- and s^- terms, the general procedure is as follows. First, rewrite Eqs.(2.7.4) and (2.7.5) in a homogenous form by introducing the new functions e^- and g^-

$$t^- = \frac{1}{2}\,(e^- + g^-) \qquad (1)$$
$$s^- = \frac{1}{2}\,(e^- - g^-) \qquad (2) \qquad\qquad (2.20)$$

They become therefore,

$$\partial^{++} e^- + \lambda f^+[\bar f^+ e^- + f^+ \bar e^-] = 0 \qquad (1)$$
$$\partial^{++} g^- + \lambda f^+[\bar f^+ g^- - f^+ \bar g^-] + 2m f^+ = 0 \qquad (2) \qquad\qquad (2.21)$$

Though one may easily derive the solution of these equations, it is interesting to note that the terms between brackets in Eqs.(2.21) are conserved quantities as a consequence of the U(1) invariance.

$$\begin{aligned} a &= \bar f^+ e^- + f^+ \bar e^- = -\overset{*}{\bar a}\,, \quad \partial^{++} a = 0 \\ b &= \bar f^+ g^- - f^+ \bar g^- = \overset{*}{\bar b}\;; \quad \partial^{++} b = 0 \end{aligned} \qquad (2.22)$$

The second step is, making use of relations (2.22), Eqs.(2.21) become linear first order differential equations easy to solve. Therefore, we have

$$\begin{aligned} t^- &= -\frac{m}{1-\lambda\rho^2} f^- = -m f^- - \frac{m\lambda\rho^2}{1-\lambda\rho^2} f^- \\ s^- &= -t^- \end{aligned} \qquad (2.23)$$

which vanish as m goes to zero as it should be. Note, moreover, that Eqs.(2.23) are not polynomial in the fields f and $\bar f$, and that the combination $F^- + G^-$ is not relevant. The B^-_μ term of Eq.(2.13.4) is obtained in a similar way by linearizing Eq.(2.7.6) and reads after some algebra as

$$B^-_\mu = 2\partial_\mu f^- - 2\lambda f^+ \partial_\mu[f^- \bar f^-] - \frac{\lambda f^-}{1-\lambda\rho^2}\left[f^i \overleftrightarrow{\partial_\mu} \bar f_i + 2i\omega' \partial_\mu \rho^2\right], \qquad (2.24)$$

where

$$\begin{aligned} f^i \overleftrightarrow{\partial_\mu} \bar f_i &= f^i \partial_\mu \bar f_i - \bar f_i \partial_\mu f^i \\ \partial_\mu \rho^2 &= f^i \partial_\mu \bar f_i + \bar f_i \partial_\mu f^i \\ \omega'(\rho^2) &= \frac{d}{d\rho^2}\,\omega(\rho^2)\,. \end{aligned} \qquad (2.25)$$

Putting now Eqs.(2.23-2.25) in the pure bosonic part of the Lagrangian (2.10) namely

$$L_B = -\frac{1}{2}\left\{ (A^-_\mu \partial^\mu \bar\varphi^+ - \bar A^-_\mu \partial^\mu \varphi^+) - 2m\varphi^+(\bar F^- - \bar G^-) + 2m\bar\varphi^+(F^- - G^-)\,. \right\} \qquad (2.26)$$

one finds after integration with respect to the harmonic variables

$$L_B = L_{GIOS} + L_\omega - \frac{2m^2\rho^2}{1-\lambda\rho^2}\,. \qquad (2.27)$$

The last term of Eq.(2.27) gives the scalar potential of the Lagrangian model (2.1). It is non-polynomial and central charge dependent. Moreover, global N= 2 supersymmetry requires the coupling parameter λ to be negative.

L_{GIOS} is the Lagrangian carried out by Galperin et al. [9] and turns out to describe the Taub-Nut metric ($\lambda < 0$). The second term depends on the extra degree of freedom ω given by Eqs.(2.18) and (2.19). These two pieces fit together and read as

$$L_{GIOS} + L_\omega = -\frac{1}{2}\left\{ \bar g_{ij}\partial_\mu f^i \partial^\mu f^j + g^{ij}\partial_\mu \bar f_i \partial^\mu \bar f_j + 2h^i{}_j \partial_\mu f^j \partial^\mu \bar f_i\,. \right\}, \qquad (2.28)$$

with

$$\bar g_{ij}[\omega'] = \frac{(2-\lambda\rho^2) - 4\omega'(\lambda\rho^2\omega' - i)}{2(1-\lambda\rho^2)}\,\lambda \bar f_i \bar f_j$$

$$g^{ij}[\omega'] = \frac{(2-\lambda\rho^2) - 4\omega'(\lambda\rho^2\omega' + i)}{2(1-\lambda\rho^2)}\,\lambda f^i f^j$$

$$h^{i}_{\cdot j}[\omega'] = -(1-\lambda\rho^2)\delta^{i}_{\cdot j} - \left[\frac{(2-\lambda\rho^2)+4\lambda\rho^2\omega'^2}{2(1-\lambda\rho^2)}\right]\lambda f^i \bar{f}_j . \quad (2.29)$$

Furthermore, it is interesting to note that choosing different "gauge conditions", one obtains different forms of the metric (2.29). The simplest gauge fixing is $\omega' = 0$ (i.e. $\omega = \frac{\text{constant}}{\lambda} = 0$ up to a global $U(1)$ transformation) lead to the GIOS metric. Another interesting gauge corresponds to the choice $\omega' = \pm 1/2$. In this case, Eqs.(2.29) reduce to

$$\bar{g}_{ij}[\pm 1/2] = \frac{[e^{\pm i\pi/4} - \sqrt{2}\lambda\rho^2]}{\sqrt{2}(1-\lambda\rho^2)} \lambda \bar{f}_i \bar{f}_j$$

$$g^{ij}[\pm 1/2] = -\frac{[e^{\mp i\pi/4} - \sqrt{2}\lambda\rho^2]}{\sqrt{2}(1-\lambda\rho^2)} - \lambda f^i f^j \quad (2.30)$$

$$h^{i}_{\cdot j}[\pm 1/2] = -(1-\lambda\rho^2)\delta^{i}_{\cdot j} - \frac{\lambda f^i \bar{f}_j}{(1-\lambda\rho^2)} .$$

What now remains to do is to examine the hyper-Kählerness of the metric (2.29). Recall that Galperin et al. [9] have shown in the case $\omega' = 0$, using an appropriate change of variables, that this particular metric reduces to the well-known hyper-Kahler Taub-Nut metric of the four-dimensional Euclidean gravitation instantons [14]. The same authors have developed, moreover, an important superfield formalism allowing them to build an SU(2) triplet of closed two-forms which is the necessary and sufficient condition for a giver metric to have a hyper-Kähler structure. In the next section, we shall use the second procedure to study the generalized metric (2.29). This is also an opportunity to see how the extra degree of freedom ω of Eq.(2.18) enters in the superfield approach.

3. HYPER-KAHLER STRUCTURE OF THE GENERALIZED GIOS METRIC

To start, let us remark that the superfield equation of motion (2.6) and its conjugate can be cast in the following SU(2) condensed form

$$\left[\tilde{D}^{++}\delta^{a}_{\cdot b} + i V^{++a}_{\cdot\ b}\right]\tilde{\Phi}^{+b} = 0 , \quad a,b = 1,2 , \quad (3.1)$$

where V^{++} is a type of a composite N=2 SYM prepotential [9] given by

$$V^{++} = \begin{pmatrix} i\lambda\tilde{\Phi}^{+}\overset{*}{\bar{\tilde{\Phi}}}{}^{+} & 0 \\ 0 & -i\lambda\tilde{\Phi}^{+}\overset{*}{\bar{\tilde{\Phi}}}{}^{+} \end{pmatrix} , \ \mathrm{Tr}\,V^{++} = 0 , \quad (3.2)$$

and $\tilde{\phi}^{+a}$ is an SU(2) isospinor defined by

$$(\tilde{\Phi}^{+a}) \equiv (\tilde{\Phi}^{+}, -\overset{*}{\bar{\tilde{\Phi}}}{}^{+})$$

$$\overline{(\tilde{\Phi}^{+a})}^{*} \equiv \left(\overset{*}{\bar{\tilde{\Phi}}}{}^{+}\right)_a = -\varepsilon_{ab}\tilde{\Phi}^{+b} ; \quad \varepsilon_{12} = -1 \quad (3.3)$$

Eq.(3.1) is similar to that of an interacting (F-S) superfield with an N=2 gauge Maxwell superfield V^{++}. This leads to introduce the bridge between the well-known λ and τ representation [8] and allows to make the transformations:

$$\tilde{\Phi}^{+} = \exp(i v) . \tilde{q}^{+} \quad (1)$$

$$V^{++} = -i\, e^{-iv} D^{++} e^{iv} \quad (2) \qquad (3.4)$$

$v = v(x,\theta^i,\bar{\theta}^i,u^{\pm}_i)$ is the bridge [8]. q^+ is a superfield depending on the central basis coordinates: $(x_\mu,\theta^i,\bar{\theta}_i,u^{\pm}_i)$ and whose first component is just $f^i(x)$

$$q^i\big|_{\theta=0} = f^i(x) \quad (3.5)$$

Eq.(3.5) is easily deduced from Eqs.(3.1) and (3.4). Moreover, the composite superfield V^{++} (3.4.2) reads in our case as

$$(V^{++}) = D^{++}v \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (3.6)$$

Making use of Eq.(3.4.2), the analyticity condition [15] $D^+\phi^+ = \frac{\partial}{\partial\theta^-}\phi^+ = 0$ becomes a covariantly one in terms of the superfield q^+, namely

$$\mathcal{D}^{+a}_{\ \cdot b}\, q^{+b} = (D^+\delta^{a}_{\cdot b} + i A^{+a}_{\ \cdot b})\, q^{+b} = 0 , \quad (3.7)$$

where the resulting (composite) spinor connection A^+ is given by

$$A^{+} = A^{i}U^{+}_{i} = -ie^{-iv}D^{+}e^{iv} \tag{3.8}$$

$$= D^{+}v \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

Now, using Eqs.(3.2) and (3.4), one can invert Eq.(3.6) and work out explicitly the bridge v in terms of $q^{\pm}$ and $\bar{q}^{\pm}$. An interesting class of solutions reads as

$$v(q,\bar{q}) = i\frac{\lambda}{2}\left(q^{+}\overset{*}{\bar{q}}{}^{-} + q^{-}\overset{*}{\bar{q}}{}^{+}\right) + \lambda C(q,\bar{q})\left[q^{+}\overset{*}{\bar{q}}{}^{-} - q^{-}\overset{*}{\bar{q}}{}^{+}\right] . \tag{3.9}$$

Or in an equivalent form compatible with notations of Sec.2

$$v(q,\bar{q}) = i\frac{\lambda}{2}\, q^{(+}\overset{*}{\bar{q}}{}^{-)} + \lambda\,\omega(q^{2}) , \tag{3.10}$$

with

$$q^{2} = q^{+}\overset{*}{\bar{q}}{}^{-} - q^{-}\overset{*}{\bar{q}}{}^{+} = q^{i}\bar{q}_{i}$$

$$\partial^{++}\omega(q^{2}) = 0 . \tag{3.11}$$

The function ω fulfil the same properties as that of Sec.2 (2.18) and might be generalized to incorporate derivatives. The pure bosonic case is obtained from (3.9)-(3.11) and (3.5) as

$$\begin{aligned} q^{2}\big|_{\theta=0} &= \rho^{2} \\ \omega(q^{2})\big|_{\theta=0} &= \omega(\rho^{2}) \\ iv\big|_{\theta=0} &= \lambda\beta \end{aligned} \tag{3.12}$$

Using Eqs.(3.8), (3.10) and the analyticity condition for the composite superfield $V^{++}(D^{+}V^{++}=0)$, Eq.(3.7) may be rewritten as

$$E^{+a}_{kb}\, D^{+}q^{kb} = 0 , \tag{3.13}$$

where $E^{+a}_{kb} = E^{ia}_{kb}(q,\bar{q})u^{+}_{i}$

$$\left(E^{ia}_{kb}\right) = \begin{pmatrix} E^{i1}_{k1} & E^{i2}_{k1} \\ E^{i1}_{k2} & E^{i2}_{k2} \end{pmatrix} = \begin{pmatrix} (1-\frac{\lambda}{2}q^{2})\delta^{i}_{k} + \frac{\lambda}{2}(q_{k}\bar{q}^{i} + 2i\omega'\bar{q}_{k}q^{i}) , & -\frac{\lambda}{2}(1+2i\omega')\bar{q}_{k}\bar{q}^{i} \\ \frac{\lambda}{2}(1-2i\omega')q_{k}q^{i} & , (1-\frac{\lambda}{2}q^{2})\delta^{i}_{k} - \frac{\lambda}{2}(\bar{q}_{k}q^{i} - 2i\omega' q_{k}\bar{q}^{i}) \end{pmatrix} \tag{3.14}$$

$E^{ia}_{kb}(q,\bar{q})$ is just the inverse of the vielbeins with the world indices kb and the tangent space indices ia. These objects have been previously used in Refs.[9,16] and, with their help, the hyper-Kähler structure of the metric becomes manifest. The determinant of this tensor can be obtained using the following formula [17]:

$$\det E = \det(AD - BD^{-1}CD) , \tag{3.15}$$

where

$$E = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

A,B,C and D are 2x2 matrices given by Eq.(3.14) and satisfying the following properties

$$\begin{aligned} \overline{D_{m}{}^{n}} &= -\varepsilon_{np}\varepsilon^{mq}A_{q}{}^{p} \\ \overline{C_{m}{}^{n}} &= \varepsilon_{np}\varepsilon^{mq}B_{q}{}^{p} \\ \det D = |D| &= (1-\lambda q^{2})\left[1 - \frac{\lambda}{2}(1+2i\omega')q^{2}\right] \\ (D^{-1})_{n}{}^{m} = (D_{m}{}^{n})^{-1} &= \left[(1-\frac{\lambda}{2}q^{2})\delta_{n}{}^{m} + \frac{\lambda}{2}(q_{n}\bar{q}^{m} - 2i\omega'\bar{q}_{n}q^{m})\right]|D|^{-1} . \end{aligned} \tag{3.16}$$

Using these relations, we have

$$\left[AD - BD^{-1}CD\right]_{k}{}^{\ell} = (1-\lambda q^{2})\left[h\,\delta_{k}{}^{\ell} + r\,\bar{q}_{k}q^{\ell}\right] \tag{3.17}$$

$$\begin{aligned} h &= 1 - \frac{\lambda}{2}(1+2i\omega')q^{2} \\ r &= \frac{\lambda}{2h}\left[4i\omega' - \frac{\lambda}{2}q^{2}(1+2i\omega')^{2}\right] . \end{aligned} \tag{3.18}$$

Finally making use of the relation

$$\det\left[AD - BD^{-1}CD\right] = (1-\lambda q^2)\, h\,(h + r q^2), \tag{3.19}$$

we get

$$\det E = (1-\lambda q^2)^3 . \tag{3.20}$$

Surprisingly enough, det E does not depend at all on the arbitrary function ω. Moreover, it should be noted that Eq.(3.13) has a freedom of rescaling the vielbeins E^{ia}_{kb} by a scalar function of q^2. As pointed out by Galperin et al., it turns out that the hyper-Kähler structure becomes manifest in terms of the following scaled vielbeins:

$$\tilde{E}^{\ ia}_{kb} = (\det E)^{-1/6} E^{\ ia}_{kb} = \frac{1}{\sqrt{1-\lambda q^2}} E^{\ ia}_{kb} . \tag{3.21}$$

The SU(2) triplet of closed two forms we are looking for reads therefore as

$$\Omega^{(ij)} = dq^{kc} \times dq^{ld}\, \tilde{E}^{\ (ia}_{kc}\, \tilde{E}^{\ j)b}_{ld}\, \varepsilon_{ab} \tag{3.22}$$

The (super)metric built from the vielbeins (3.14) is given by

$$G_{kc,ld} = \tilde{E}^{\ ia}_{kc}\, \tilde{E}^{\ jb}_{ld}\, \varepsilon_{ij}\, \varepsilon_{ab} . \tag{3.23}$$

The pure bosonic part of Eq.(3.23) reads then

$$\left(G_{ia,jb}\right) \equiv \begin{pmatrix} \dfrac{(2-\lambda\rho^2) - 4\omega'(\lambda\rho^2\omega' - i)}{2(1-\lambda\rho^2)}\lambda \bar{f}_i \bar{f}_j & , & -(1-\lambda\rho^2)\varepsilon_{ij} + \dfrac{(2-\lambda\rho^2) + 4\lambda\rho^2\omega'^2}{2(1-\lambda\rho^2)}\lambda \bar{f}_i f_j \\ -(1-\lambda\rho^2)\varepsilon_{ij} - \dfrac{(2-\lambda\rho^2) + 4\lambda\rho^2\omega'^2}{2(1-\lambda\rho^2)}\lambda f_i \bar{f}_j & , & \dfrac{(2-\lambda\rho^2) - 4\omega'(\lambda\rho^2\omega' + i)}{2(1-\lambda\rho^2)}\lambda f_i f_j \end{pmatrix} \tag{3.24}$$

which should be compared to that given by Eqs.(2.29).

To conclude this study on the pure bosonic part of Eq.(2.1), we would like to emphasize that the generalized solutions of the auxiliary field Eqs.(2.13) leading to the metric (2.29) are compatible with the hyper-Kähler structure of

N=2, d=4 supersymmetric theories. These solutions depend on the extra degree of freedom which might have a geometrical meaning. This feature is expected to be present in all hyper-Kähler metric building from the harmonic superspace techniques. Furthermore, we also learnt that the induced scalar potential is non-polynomial in the field ρ^2 and has a simple form. This potential, however, is ω-independent contrary to the metric. The interpretation of the extra degree ω and its underlying symmetry is postponed to a future work.

4. SPINORIAL INTERACTIONS

In this section, we give the spinorial contributions by solving the auxiliary field equations (2.7) in presence of fermionic fields. The procedure is very similar to the bosonic case of Sec.2. The main ingredient is the U(1) global symmetry of the Lagrangian (2.1) leading to the following factorizations:

$$\begin{aligned} \varphi^+ &= f^+ \exp -\lambda\beta \\ \Psi &= \psi\, e^{-\lambda\beta}, \quad X = \chi\, e^{\lambda\beta} \\ F^- &= t^- e^{-\lambda\beta}, \quad G^- = s^- e^{-\lambda\beta} \\ A^-_\mu &= B^-_\mu\, e^{-\lambda\beta} \end{aligned} \tag{4.1}$$

where $\beta = \beta(x_A, u^\pm_i)$ is given by Eq.(2.18). The unknown functions ψ, χ, t^-, s^- and B^-_μ are obtained from Eqs.(2.7). Using relations (2.3) and (2.12) the constraints (2.7) can be linearized in the fields and one works out easily their solutions. For the spinor fields ψ and χ, we find

$$\begin{aligned} \psi(x_A, u^\pm) &= \frac{1}{1-\lambda\rho^2}\left[(1-\lambda f^+\bar{f}^-)\xi + \lambda f^+ f^- \eta\right] \\ \chi(x_A, u^\pm) &= \frac{1}{1-\lambda\rho^2}\left[-\lambda \bar{f}^+\bar{f}^-\xi + (1+\lambda f^-\bar{f}^+)\eta\right] \end{aligned} \tag{4.2}$$

where $\xi = \xi(x_A)$ and $\eta = \eta(x_A)$ are two Weyl spinors satisfying the constraints

$$\partial^{++}\xi = \partial^{++}\eta = 0 . \tag{4.3}$$

Note that the solutions (4.2) depend on the bosonic fields f^i and $\bar{f}_i$ as well as on two arbitrary independent Weyl spinors ξ and η. These fermions coincide respectively with ψ and χ in the limit λ goes to zero. Moreover, it is

interesting to note that the solutions ψ and χ are rotated by a local $GL(2,\mathbb{C})$ transformation given by

$$\begin{pmatrix} \psi \\ \chi \end{pmatrix} = \frac{1}{(1-\lambda\rho^2)} \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} \tag{4.4}$$

or formally by defining $(\Gamma^\alpha) \equiv (\chi, -\psi)$ and $(\Lambda^\alpha) \equiv (\eta, -\xi)$ $\alpha = 1,2$.

$$\Gamma_\alpha = \mathbb{U}_\alpha{}^\sigma \Lambda_\sigma \tag{4.5}$$

where

$$\begin{aligned} a &= a_i{}^k u^{+i} u^-_k \;; \quad b = b_i{}^k u^{+i} u^-_k \\ a_{ij} &= \varepsilon_{ij} + \lambda f_i \bar{f}_j = (1 - \tfrac{\lambda}{2}\rho^2)\varepsilon_{ij} + \tfrac{\lambda}{2} f_{(i}\bar{f}_{j)} \\ b_{ij} &= -\lambda f_i f_j = \tfrac{\lambda}{2} f_{(i} f_{j)} \\ \det \mathbb{U} &= \frac{1}{1-\lambda\rho^2} . \end{aligned} \tag{4.6}$$

a, b and det U go respectively to the values 1, 0 and 1 as λ goes to zero. Demanding to the solutions (4.2) to reproduce the right Dirac Lagrangian coefficients once plugged in Eq.(2.1), one should have the following normalized solutions:

$$\begin{aligned} \psi(x_A, u^\pm) &= \frac{1}{\sqrt{1-\lambda\rho^2}} \left[(1 - \lambda f^+ \bar{f}^-)\xi + \lambda f^+ f^- \eta \right] \\ \chi(x_A, u^\pm) &= \frac{1}{\sqrt{1-\lambda\rho^2}} \left[-\lambda \bar{f}^+ \bar{f}^- \xi + (1 + \lambda f^- \bar{f}^+)\eta \right] \end{aligned} \tag{4.7}$$

or equivalently in the condensed form as in Eq.(4.5)

$$\Gamma_\alpha = \mathbb{K}_\alpha{}^\sigma \Lambda_\sigma \;; \quad \mathbb{K}_\alpha{}^\sigma = \mathbb{K}_{i\alpha}{}^{j\sigma} u^{+i} u^-_j \tag{4.8}$$

$$\left(\mathbb{K}_{i\alpha}{}^{j\sigma}\right) \equiv \begin{pmatrix} \mathbb{K}_{i1}{}^{j1} & \mathbb{K}_{i1}{}^{j2} \\ \mathbb{K}_{i2}{}^{j1} & \mathbb{K}_{i2}{}^{j2} \end{pmatrix} = \begin{pmatrix} a_i{}^j & b_i{}^j \\ -\bar{b}_i{}^j & \bar{a}_i{}^j \end{pmatrix} \frac{1}{\sqrt{1-\lambda\rho^2}} \tag{4.9}$$

$$\det \mathbb{K} = 1 .$$

As a consequence, the local symmetry (4.5) rotating the spinor solutions becomes an SU(2) symmetry.

Furthermore, it is interesting to note that the normalization of Eq.(4.7) is the same as the vielbein one, $(\det E)^{-1/6}$, encountered in the previous section Eqs.(3.20) and (3.21). In fact, there is a striking similarity between the 2x2 matrix $\mathbb{K}$ of Eq.(4.8) and the vielbeins of Eq.(3.14). They are deduced one from the other by making the following correspondances

$$\begin{aligned} \mathbb{U} &\longleftrightarrow E \\ f_{(i}\bar{f}_{j)} &\longleftrightarrow f_i\bar{f}_j + 2i\omega' \bar{f}_i f_j \\ f_{(i} f_{j)} &\longleftrightarrow f_i f_j + 2i\omega' f_i f_j \end{aligned} \tag{4.10}$$

Following the same strategy as in the previous cases, one can derive the remaining unknown functions of Eq.(4.1). They are spinor- dependent solutions and read after some straightforward algebra as

$$\begin{aligned} t^- = &- \frac{m}{(1-\lambda\rho^2)} f^- - \frac{(2 - \lambda f^+\bar{f}^-) - \lambda\rho^2(1 - \lambda f^+\bar{f}^-)}{(1-\lambda\rho^2)^2} (\lambda f^-)\xi\eta \\ &+ \frac{2 - \lambda f^+\bar{f}^-}{2(1-\lambda\rho^2)} (\lambda\bar{f}^-)\xi\xi - \frac{\lambda f^+ f^-}{2(1-\lambda\rho^2)} (\lambda f^-)\eta\eta \;; \end{aligned} \tag{4.11}$$

$$\begin{aligned} s^- = &\frac{m}{(1-\lambda\rho^2)} f^- + \frac{(2 - \lambda f^+\bar{f}^-) - \lambda\rho^2(1 - \lambda f^+\bar{f}^-)}{(1-\lambda\rho^2)^2} (\lambda f^-)\bar{\xi}\bar{\eta} \\ &+ \frac{(2 - \lambda f^+\bar{f}^-)}{2(1-\lambda\rho^2)} (\lambda\bar{f}^-)\bar{\eta}\bar{\eta} - \frac{\lambda f^+ f^-}{2(1-\lambda\rho^2)} (\lambda f^-)\bar{\xi}\bar{\xi} \end{aligned} \tag{4.12}$$

Note that t^- and s^- are no longer proportional as in Eq.(2.23). t^- depends on ξ and η whereas s^- depends on their conjugates $\bar{\xi}$ and $\bar{\eta}$. These fermionic contribution parts, however, are related by the following change of variables:

$$\begin{aligned} \xi &\longleftrightarrow \bar{\eta} \\ \eta &\longleftrightarrow -\bar{\xi} \end{aligned} \tag{4.13}$$

Such a symmetry becomes more clear, if one uses the combinations $e^- = t^- + s^-$ and $g^- = t^- - s^-$. Finally, the B^-_μ solution reads as

$$B^-_\mu = 2\partial_\mu f^- - 2\lambda f^+ \partial_\mu [f^- \bar{f}^-] - \frac{\lambda f^-}{1-\lambda\rho^2}\left[f^i \overleftrightarrow{\partial_\mu} \bar{f}_i + 2i\partial_\mu \omega\right] + \frac{i}{(1-\lambda\rho^2)} Q^-_\mu . \tag{4.14}$$

where the fermionic contribution Q^-_μ is given by

$$Q^-_\mu = \frac{\lambda f^-}{(1-\lambda\rho^2)}\left[(2-\lambda f^+\bar{f}^-) - \lambda\rho^2(1-\lambda f^+\bar{f}^-)\right](\xi\sigma_\mu\bar{\xi} - \eta\sigma_\mu\bar{\eta}) + \lambda\bar{f}^-\left[2-\lambda f^+\bar{f}^-\right]\xi\sigma_\mu\bar{\eta} + \lambda f^-\left[\lambda f^+ f^-\right]\eta\sigma_\mu\bar{\xi} . \tag{4.15}$$

Therefore, the full set of auxiliary fields reduces to a simple expression depending on the N=2 on shell multiplet degrees of freedom (f^i,ξ,η) and their conjugates. To evaluate the fermionic contribution of the Lagrangian (2.8), it is interesting to calculate the following quantities

$$\begin{aligned} Z_\mu &= \bar{f}^+ B^-_\mu + f^+ \bar{B}^-_\mu - iJ_\mu \\ &= \partial_\mu f^{(+}\bar{f}^{-)} + \frac{1}{(1-\lambda\rho^2)}\left[f^i \overleftrightarrow{\partial_\mu} \bar{f}_i + 2i\lambda\rho^2\partial_\mu\omega - i(\xi\sigma_\mu\bar{\xi} - \eta\sigma_\mu\bar{\eta})\right] \end{aligned} \tag{4.16}$$

and

$$\begin{aligned} k &= \bar{f}^+ t^- + f^+ \bar{s}^- + \psi\chi \\ &= \frac{m\rho^2}{1-\lambda\rho^2} + \frac{1}{(1-\lambda\rho^2)^2}\xi\eta \end{aligned} \tag{4.17}$$

$$\begin{aligned} \ell &= \bar{f}^+ s^- + f^+ \bar{t}^- - \bar{\psi}\bar{\chi} \\ &= -\frac{m\rho^2}{1-\lambda\rho^2} - \frac{1}{(1-\lambda\rho^2)^2}\bar{\xi}\bar{\eta} , \end{aligned} \tag{4.18}$$

and finally

$$\begin{aligned} \zeta^+ &= \bar{f}^+\psi - f^+\chi \\ &= \frac{1}{\sqrt{1-\lambda\rho^2}}\left[\bar{f}^+\xi - f^+\eta\right] \end{aligned} \tag{4.19}$$

Note that these quantities satisfy the following constraints

$$\begin{aligned} \partial^{++} Z_\mu &= 2\partial_\mu f^+ \bar{f}^+ \\ \partial^{++} k &= \partial^{++}\ell = \partial^{++}\zeta^+ = 0 \end{aligned} \tag{4.20}$$

Putting now the relations (4.1) and (4.7)-(4.15) in the Lagrangian (2.10) one obtains a quantity depending on the fields f^i,ξ,η and their conjugates and at most eight harmonic variables. Then, using Eqs.(4.16)-(4.20) and the $u^\pm_i$ integration rules [8,9]

$$\int du\, (u^+)^{(m}(u^-)^{n)}(u^+)_{(k}(u^-)_{\ell)} = \begin{cases} \dfrac{(-)^n m!\, n!}{(m+n+1)!}\, \delta^{(i_1}_{(j_1} \cdots \delta^{i_{m+n})}_{j_{k+\ell})} & \text{if } \begin{matrix} m=\ell \\ n=k \end{matrix} \\ 0 & \text{otherwise} \end{cases}$$

$$(u^+)^{(m}(u^-)^{n)} \equiv u^{+(i_1} \cdots u^{+i_m} u^{-j_1} \cdots u^{-j_n)} , \tag{4.21}$$

One finds after a tedious but straightforward algebra

$$2L = L_{bosonic} + L_{fermionic} + L_m + V(\rho^2,m,\lambda) \tag{4.22}$$

where $L_{bosonic}$ is given by Eq.(2.28) and

$$L_m = -4m^2\rho^2 + 2m(\xi\eta + \bar{\xi}\bar{\eta}) \tag{4.23}$$

$$V(\rho^2,m,\lambda) = -4m^2\frac{\lambda\rho^4}{(1-\lambda\rho^2)} \tag{4.24}$$

The fermionic part of the Lagrangian can be put in the following form

$$L_{\text{fermionic}} = L_{KC} + L_S \quad . \tag{4.25}$$

The covariant kinetic term L_{KC} is given by

$$\begin{aligned} L_{KC} = & -2i\left(\partial_\mu \xi \sigma^\mu \bar{\xi} - \eta \sigma^\mu \partial_\mu \bar{\eta}\right) \\ & -\frac{i}{3}\, \varepsilon^{ab} \mathcal{H}_{ia}{}^{(ic} \partial_\mu \mathcal{H}_{jb}{}^{j)d} \Lambda_c \sigma^\mu \bar{\Lambda}_d \\ & +\frac{2i}{6(1-\lambda\rho^2)}\left[A_\mu\left(\xi\sigma^\mu\bar{\xi} - \eta\sigma^\mu\bar{\eta}\right) + B_\mu \xi\sigma^\mu\bar{\eta} + \bar{B}_\mu \eta\sigma^\mu\bar{\xi}\right] . \end{aligned} \tag{4.26}$$

with

$$\begin{aligned} A_\mu = & \left[2i\omega'\partial_\mu\rho^2 \delta^i_{(l}\delta^j_{m)} + \frac{1}{8}\partial_\mu\left(a_k{}^n - \bar{a}_k{}^n\right)\delta^i_{(l}\delta^j_m\delta^k_{n)}\right]\left[a_i{}^l \bar{a}_j{}^m - b_i{}^l b_j{}^m\right] \\ & + \frac{\lambda\rho^2}{10(1-\lambda\rho^2)}\left[10 - 35\lambda\rho^2 + 6(\lambda\rho^2)^2 + 4(\lambda\rho^2)^3\right] f^i \overleftrightarrow{\partial}_\mu \bar{f}_i \\ & + \frac{i\lambda\rho^2}{10(1-\lambda\rho^2)}\left[20 + 10\lambda\rho^2 - 40(\lambda\rho^2)^2\right]\partial_\mu\omega \\ & - \frac{i}{10(1-\lambda\rho^2)}\left[15 - 105\lambda\rho^2 + 155(\lambda\rho^2)^2 - 85(\lambda\rho^2)^3 + 41(\lambda\rho^2)^4 + 24(\lambda\rho^2)^5\right] \times \\ & \qquad \left(\xi\sigma_\mu\bar{\xi} - \eta\sigma_\mu\bar{\eta}\right) , \end{aligned} \tag{4.27}$$

and

$$\begin{aligned} B_\mu = & \; 2\, a_i{}^l \bar{b}_j{}^m \left[2i\omega'\partial_\mu\rho^2 \delta^i_{(l}\delta^j_{m)} + \frac{1}{8}\partial_\mu\left(a_k{}^n - \bar{a}_k{}^n\right)\delta^i_{(l}\delta^j_m\delta^k_{n)}\right] \\ & - \frac{1}{2}\left[14 - 14\lambda\rho^2 - 9(\lambda\rho^2)^2 - 2(\lambda\rho^2)^3\right]\bar{f}_i \partial_\mu \bar{f}^i \\ & - \frac{i\lambda\rho^2}{20(1-\lambda\rho^2)}\left[60 - 45\lambda\rho^2 + 50(\lambda\rho^2)^2 - 14(\lambda\rho^2)^3\right]\eta\sigma_\mu\bar{\xi} \quad . \end{aligned} \tag{4.28}$$

The quadratic and the quartic spinor couplings of Eq.(4.25) read as

$$\begin{aligned} L_s = & \frac{1}{3(1-\lambda\rho^2)^2}\Big\{ mX\left(\xi\eta + \bar{\xi}\bar{\eta}\right) + \\ & + \lambda Y\left[(\xi\xi)(\bar{\xi}\bar{\xi}) + (\eta\eta)(\bar{\eta}\bar{\eta})\right] + \lambda Z(\xi.\eta)(\bar{\xi}.\bar{\eta})\Big\} \end{aligned} \tag{4.29}$$

with

$$\begin{aligned} X &= \left[-18\lambda\rho^2 + 14(\lambda\rho^2)^2 - 7(\lambda\rho^2)^3\right] \\ Y &= \left[-\frac{1}{10}\lambda\rho^2\left(30 - 15\lambda\rho^2\right)\right] \\ Z &= \frac{4}{10}\left[30 - 60\lambda\rho^2 + 40(\lambda\rho^2)^2 - 17(\lambda\rho^2)^3\right] \end{aligned} \tag{4.30}$$

This ends the computation of the fermionic contribution.

5. CONCLUSION

In this paper,, we gave a generalization of the GIOS metric by working out more general solutions of the auxiliary field equations of motion. The resulting generalized metric depends on an extra degree of freedom which can be eliminated through a gauge fixing condition. The later may be chosen to incorporate the effects of central charges.

Then, using the vielbein formalism, we checked the hyper-Kähler structure of the generalized metric by constructing a real triplet of closed two forms. The closure of these two forms goes straightforwardly and turns out to give no constraint of the extra degree of freedom ω. Furthermore, we found that the vielbein normalization $(\det E)^{-1/6}$ given by Galperin et al. is preserved since it remains ω independent.

Finally, we examined the fermionic contributions. We gave the full set of solutions of the auxiliary field equations of motion in presence of spinor fields. For the fermionic fields, we showed that they depend on the bosonic fields as it should be. These solutions turn out to be rotated by a local SU(2) symmetry leading to the right normalization of the spinor field which is identical to the vielbein one. The complete Lagrangian, in terms of the component fields of the N=2 on shell multiplet (f^i, ξ, η) is worked out explicitly.

ACKNOWLEDGMENTS

The authors (A.E.H., T.L. and E.H.S.) would like to thank Professor Abdus Salam, the International Atomic Energy Agency and UNESCO for hospitality at the International Centre for Theoretical Physics, Trieste, where part of this work has been done.

APPENDIX

We collect below some useful relations for checking the closure of the triplet of two forms

$$
\begin{aligned}
q_{[i}\bar q_{j]} &= q_i \bar q_j - q_j \bar q_i = q^2 \varepsilon_{ij} \; ; \; q^2 = q^i \bar q_i \\
q^i \bar q_j - \bar q^i q_j &= q^2 \delta^i_j \\
\bar q_{[m} \delta_{k]}^{(i} \delta^{j)}_l &= \bar q^{(i} \delta^{j)}_l \varepsilon_{mk} \\
\bar q^{(i} \delta^{j)}_{[m} \varepsilon_{k]l} &= \bar q^{(i} \delta^{j)}_l \varepsilon_{km} \\
\bar q_{[m} \delta_{k]}^{(i} \bar q^{j)} &= 2 \bar q^i \bar q^j \varepsilon_{mk}
\end{aligned}
\tag{A.1}
$$

The following relations are useful when integrating out the harmonic variables

$$
\begin{aligned}
&\int du\, \lambda^2 f^{+2} \bar f^{-2} = \int du\, \lambda^2 f^{-2} \bar f^{+2} = -\int du\, \lambda^2 f^+ f^- \bar f^+ \bar f^- = \frac{1}{6} (\lambda p^2)^2 \\
&\lambda^3 \int du\, f^+ f^{-2} \bar f^- \bar f^{+2} = -\lambda^3 \int du\, f^- f^{+2} \bar f^+ \bar f^{-2} = -\frac{1}{12} (\lambda p^2)^3 \\
&\lambda^3 \int du\, f^{+3} \bar f^{-3} = -\lambda^3 \int du\, f^{-3} \bar f^{+3} = -\frac{1}{4} (\lambda p^2)^3 \\
&\lambda^4 \int du\, f^{+2} f^{-2} \bar f^{+2} \bar f^{-2} = \frac{1}{30} (\lambda p^2)^4 \\
&\lambda^4 \int du\, f^+ f^{-3} \bar f^- \bar f^{+3} = -\frac{1}{20} (\lambda p^2)^4 \\
&\lambda \int du\, f^+ \bar f^+ \bar f^- \partial_\mu \bar f^- = \lambda \int du\, \bar f^+ \bar f^- f^- \partial_\mu \bar f^+ = \frac{1}{2} (\lambda p^2)\, \bar f^i \partial_\mu \bar f_i \\
&\lambda \int du\, f^+ f^- \partial_\mu f^{(-} \bar f^{+)} = \frac{1}{3} (\lambda p^2)\, \bar f^i \partial_\mu \bar f_i \\
&\lambda \int du\, \bar f^{+2} \bar f^- \partial_\mu \bar f^- = \int du\, \bar f^{-2} \bar f^+ \partial_\mu f^+ = 0 \\
&\lambda^2 \int du\, f^+ f^- \bar f^+ \bar f^{-2} \partial_\mu f^+ = -\frac{1}{12} (\lambda p^2)^2\, \bar f_k \partial_\mu f^k \\
&\lambda^2 \int du\, f^+ f^- \bar f^{+2} \bar f^- \partial_\mu f^- = \frac{1}{12} (\lambda p^2)^2\, \bar f_k \partial_\mu f^k \\
&\lambda^2 \int du\, f^{+2} \bar f^- \bar f^+ \partial_\mu \bar f^- = \lambda^2 \int du\, f^+ f^- \bar f^+ \bar f^{-2} \partial_\mu \bar f^+ = \frac{1}{12} (\lambda p^2)^2\, \bar f^i \partial_\mu \bar f_i \\
&\lambda^2 \int du\, \bar f^+ \bar f^{-3} f^+ \partial_\mu f^+ = 0
\end{aligned}
\tag{A.2}
$$

REFERENCES AND FOOTNOTE

[1] B. Zumino, Phys. Lett. B87, 203 (1979).

[2] L. Alvarez-Gaumé and D.Z. Freedman, Comm. Math. Phys. 80, 443 (1981); J. Bagger, SLAC, preprint, SLAC.PUB.3461 (September 1984).

[3] S.J. Gates, C.M. Hull and M. Rocek. Nucl. Phys. B248, 157 (1984); S.J. Gates, Nucl. Phys. B248, 349 (1984); S. Mukhi in Procs. of the Summer Workshop on High Energy Physics and Cosmology, ICTP, Trieste, June 1986 and references therein.

[4] D. Gross, J. Harvey, E. Martinec and R. Rohm, Nucl. Phys. B256, 253 (1985); Nucl. Phys. B267, 75 (1986); J.H. Schwarz, Phys. Rep. 89C, 223 (1982); CALTECH, preprint CALT/68/1290 (1985); L. Brink, CERN, preprint TH-4006/84 (1984); lecture delivered at the NATO Advanced Study Institute on Supersymmetry, Bonn, 20-31 August 1984.

[5] J.H. Schwarz, M. Green and E. Witten, Superstring Theories (Cambridge University Press, 1986).

[6] U. Lindstrom and M. Rocek, Nucl. Phys. B222, 285 (1983); R. Brook, F. Muhammad and S.J. Gates, Nucl. Phys. B268, 599 (1986).

[7] A. Karlhede, U. Lindstrom and M. Rocek, Comm. Math. Phys. 108, 529 (1987); A.Y. Morozov, A.M. Perelomov and M.A. Shifman, Nucl. Phys. B248, 279 (1984); N.J. Hitchin, A. Karlhede, U. Lindstrom and M. Rocek, Comm. Math. Phys. 108, 535 (1987); J. Bagger and E. Witten, Nucl. Phys. B222, 1 (1983).

[8] A. Galperin, E. Ivanov, S. Kalitzin, V. Ogievetsky and E. Sokatchev, Class. Quant. Grav. 1, 469 (1984).

[9] A. Galperin, E. Ivanov, V. Ogievetsky and E. Sokatchev, Comm. Math. Phys. 103, 515 (1986).

[10] T. Lhallabi and E.H. Saidi, "The (4,0) and (4,4) supersymmetric non-linear σ models in the $D=2$ harmonic superspace", Rabat L.P.T., preprint (September 1986), to appear in Int. J. Mod. Phys. A.

[11] E. Sokatchev and K. Stelle, Class. Quant. Grav. 3, 501 (1987).

[12] D. Olivier and G. Valent, L.P.T.H.E., preprint 86/49 (December 1986)

[13] E.H. Saidi, Int. J. Mod. Phys. 2, 695 (1987); A.E. Hassouni, T. Lhallabi, E.G. Oudrhiri –Safiani and E.H. Saidi, Rabat preprint, L.P.T./4/87, (April 1987), submitted to Int. J. Mod. Phys. A.

[14] T. Eguchi, P. Gilkey and A. Hanson, "Gravitation, gauge theories and differential geometry", Phys. Rep. 66C, 213 (1980).

[15] In fact, there is another analycity condition given by $\bar{D}^{+}\phi^{+} = 0$. It leads to the same vielbein.

[16] G. Sierra and P.K. Townsend, Phys. Lett. 124B, 497 (1983); Nucl. Phys. B233, 289 (1984).

[17] A.Yu. Morozov and A. Perelomov, Nucl. Phys. B271, 620 (1986).